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AMALGAMATION AND INTERPOLATION IN THE CATEGORY OF HEYTING ALGEBRAS

A.M. PITTS
Department of Pure Mathematics, University of Cambridge, England

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Introduction

This is the first of two papers describing how properties of open continuous maps between locales (which are the lattice-theoretic generalisation of topological spaces) can be used to give very straight-forward, constructive proofs of certain properties of first-order intuitionistic theories. The properties we have in mind are those of stability of a conservative interpretation of theories under pushout, and appropriate categorical formulations of Craig's Interpolation Theorem and the Beth Definability Theorem. It is thus the methods of proof rather than the results themselves that are novel, and we present them in the spirit of a demonstration of the usefulness of a category-theoretic approach to constructive logic.

In this paper we will consider only *propositional* intuitionistic theories and their lattice-theoretic counterpart, Heyting algebras. At this level the Interpolation Theorem becomes a statement about free Heyting algebras:

Theorem. *Given a set X, let $F(X)$ denote the free Heyting algebra on X. If $\phi \in F(X)$, $\psi \in F(Y)$ and $\phi \leq \psi$ in $F(X \cup Y)$, then there is $\theta \in F(X \cap Y)$ with $\phi \leq \theta$ in $F(X)$ and $\theta \leq \psi$ in $F(Y)$.* □

The theorem asserts that the pushout square

$$\begin{array}{ccc} F(X) & \hookrightarrow & F(X \cup Y) \\ \uparrow & & \uparrow \\ F(X \cap Y) & \hookrightarrow & F(Y) \end{array}$$

in the category **Ha** of Heyting algebras and morphisms, has the "interpolation property", which we may define in general as follows:

Definition. Let

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle k} \\ A & \xrightarrow{g} & C \end{array}$$

be a commutative square of partially ordered sets and order-preserving maps. We say that it has the *interpolation property* iff for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$, then there is $a \in A$ with $b \leq f(a)$ and $g(a) \leq c$.

Remark. In the case that f and k have left adjoints $f_!$ and $k_!$ respectively, then the commutative square has the interpolation property iff it satisfies a "Beck–Chevalley condition", namely

$$k_! \circ h = g \circ f_! .$$

We shall prove below

Theorem B. *Every pushout square in* **Ha** *has the interpolation property.*

Now it is known (cf. [4]) that there is an intimate connection between the Interpolation Theorem and the *amalgamation property*, which in this context says that if $f : A \rightarrowtail B$ and $g : A \rightarrowtail C$ are monomorphisms in **Ha**, then there is a Heyting algebra D and monomorphisms $h : B \rightarrowtail D$ and $k : C \rightarrowtail D$ with $h \circ f = k \circ g$. More generally, using properties of open maps of locales and some simple considerations on filters and ideals, we prove

Theorem A. *Monomorphisms are stable under pushout in* **Ha**, *i.e. if*

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle k} \\ A & \xrightarrow{g} & C \end{array}$$

is a pushout square in **Ha** *and f is a monomorphism, so is k.*

Then Theorem B follows from Theorem A together with elementary properties of quotients of Heyting algebras. In the final section of the paper we make some remarks concerning the analogues of these theorems for the coherent fragment of intuitionistic propostional logic (distributive lattices) and for geometric propositional logic (frames).

The lattice-theoretic methods used in this paper are all constructively valid. Apart from any intrinsic value this approach may have, it becomes essential in the sequel [5], where for example the proofs require the application of properties of open maps of locales defined in toposes other than the base topos of (possibly classical) sets. In particular, no use is made of the Prime Ideal Theorem, and we deal with lattices of ideals rather than spaces of prime ideals.

We adopt the convention that a partially ordered set is a lattice if it has *all* finite meets and joins including the empty ones, i.e. lattices will always have top and bottom elements, denoted $\top$ and $\bot$ respectively.

1. Frames, locales and open maps

A *frame* is a complete lattice A in which binary meets distribute over arbitrary joins:

$$a\wedge\bigvee S=\bigvee\{a\wedge s\mid s\in S\}\quad\text{for all } a\in A, S\subseteq A.$$

A morphism of frames is a map preserving finite meets and arbitrary joins; let **Frm** denote the category of frames and frame morphisms. Then the category **Loc** of *locales* is just the opposite category, $\mathbf{Frm}^{\mathrm{op}}$. A morphism $f: A\to B$ in **Loc** is called a *continuous map of locales*; the corresponding frame morphism is conventionally denoted by $f^*: B\to A$ and its right adjoint by $f_*: A\to B$. An introduction to the theory of locales and its relation to general topology may be found in [1], whilst the reader should refer to [2] for a full exposition of the particular notion we need, namely that of an open continuous map between locales. (*Warning:* the terminology of [2] is non-standard; frames are there called locales and locales called spaces.)

Definitions. Suppose $f: B\to A$ is a morphism of meet semilattices with a left adjoint $f_!: A\to B$. The adjoint is said to satisfy *Frobenius reciprocity* iff for all a in A and b in B

$$f_!(a\wedge f(b))=f_!(a)\wedge b.$$

We then define a continuous map $f: A\to B$ of locales to be *open* iff $f^*: B\to A$ has a left adjoint $f_!: A\to B$ satisfying Frobenius reciprocity.

Since a frame is in particular a Heyting algebra, and

$$\begin{array}{ccc} A & \xrightarrow{\;f_!\;} & B \\ {\scriptstyle (-)\wedge f^*b}\big\downarrow & & \big\downarrow{\scriptstyle (-)\wedge b} \\ A & \xrightarrow{\;f_!\;} & B \end{array}$$

commutes iff the corresponding square of right adjoints

$$\begin{array}{ccc} A & \xleftarrow{f^*} & B \\ {\scriptstyle f^*b\to(-)}\uparrow & & \uparrow{\scriptstyle b\to(-)} \\ A & \xleftarrow{f^*} & B \end{array}$$

commutes, we see that $f: A \to B$ is open iff f^* preserves arbitrary meets and Heyting implication. There is thus a contravariant forgetful functor from the category **OLoc** of locales and open continuous maps to the category **Ha** of Heyting algebras, sending $f: A \to B$ to $f^*: B \to A$. Finally, recall that a continuous map $f: A \to B$ of locales is a *surjection* iff $\mathrm{id}_B = f_* \circ f^*$, or equivalently iff f^* is a monomorphism (of frames). Note that if f is open, it is also a surjection iff $f_!(\top) = \top$.

The property of open maps we need is the following:

1.1. Theorem. *Suppose that*

$$\begin{array}{ccc} P & \xrightarrow{q} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{f} & A \end{array}$$

is a pullback square in the category **Loc** *with f open. Then q is also open. If furthermore f is a surjection, then so is q.*

Proof. We shall sketch the proof; full details may be found in Chapter V of [2].

Working in the category **Frm**, the pushout of $f^*: A \to B$ along $g^*: A \to C$ may be constructed as a tensor product, $P = B \otimes_A C$. As a complete lattice this is generated by elements $b \otimes c$ $(b \in B, c \in C)$ subject to the relations

$$(\bigvee R) \otimes C = \bigvee \{ r \otimes c \mid r \in R \} \qquad (R \subseteq B),$$

$$b \otimes (\bigvee S) = \bigvee \{ b \otimes s \mid s \in S \} \qquad (S \subseteq C)$$

and

$$(b \wedge f^* a) \otimes c = b \otimes (g^* a \wedge c) \qquad (a \in A, b \in B, c \in C).$$

Then $p^*: B \to B \otimes_A C$ and $q^*: C \to B \otimes_A C$ are the maps

$$b \mapsto b \otimes \top \quad \text{and} \quad c \mapsto \top \otimes c$$

respectively.

If f^* has a left adjoint $f_!$ satisfying Frobenius reciprocity, then we get a well-defined map $q_!: B \otimes_A C \to C$ defined on generators by

$$b \otimes c \mapsto g^*(f_! b) \wedge c.$$

Furthermore $q_!$ is left adjoint to q^* and satisfies Frobenius reciprocity. Thus q is open when f is.

If we also have that f is surjective, i.e. $f_!(\top)=\top$, then

$$q_!(\top)=q_!(\top\otimes\top)=g^*(f_!\top)\wedge\top=g^*(\top)=\top,$$

so that q is also surjective. □

2. The locale of ideals of filters

If D is a distributive lattice, let $\mathscr{I}(D)$ denote the set of ideals of D partially ordered by inclusion and $\mathscr{F}(D)$ denote the set of filters of D partially ordered by *reverse* inclusion. (Thus $\mathscr{F}(D)=(\mathscr{I}(D^{\mathrm{op}}))^{\mathrm{op}}$.) $\mathscr{I}(D)$ and $\mathscr{F}(D)$ are both complete, distributive lattices and indeed $\mathscr{I}(D)$ is a frame, although $\mathscr{F}(D)$ is not in general. ($\mathscr{I}(D)$ is the typical *coherent* locale: cf. [1].) There are order-preserving, injective maps

$$\downarrow_D : D\to\mathscr{I}(D) \quad\text{and}\quad \uparrow_D : D\to\mathscr{F}(D)$$

assigning principal ideals and filters respectively. $\downarrow_D$ preserves arbitrary meets and finite joins; dually $\uparrow_D$ preserves finite meets and arbitrary joins.

In a meet semilattice A, the Heyting implication of two elements a_1,a_2, if it exists is the unique element $a_1\to a_2$ satisfying

$$a\wedge a_1\le a_2 \;\Leftrightarrow\; a\le a_1\to a_2 \quad\text{for all } a\in A.$$

We then say that a morphism $f:A\to B$ of meet semilattices *preserves implications* iff whenver $a_1\to a_2$ exists in A, $f(a_1)\to f(a_2)$ exists in B and equals $f(a_1\to a_2)$. We then have:

2.1. Lemma. *If D is a distributive lattice, $\downarrow_D : D\to\mathscr{I}(D)$ and $\uparrow_D : D\to\mathscr{F}(D)$ both preserve implications.* □

We can extend the assignments

$$D\mapsto\mathscr{I}(D) \quad\text{and}\quad D\mapsto\mathscr{F}(D)$$

to functors on the category of distributive lattices and order-preserving maps to itself, as follows. Given $f:D\to D'$ define

$$\mathscr{I}(f):\mathscr{I}(D)\to\mathscr{I}(D')$$

by sending an ideal $I\subseteq D$ to $\mathscr{I}f(I)=\{d'\in D' \mid \exists d\in I\; d'\le f(d)\}$, the ideal generated by the image of I under f. Similarly define

$$\mathscr{F}(f):\mathscr{F}(D)\to\mathscr{F}(D')$$

by sending a filter $\delta\subseteq D$ to $\mathscr{F}f(\delta)=\{d'\in D' \mid \exists d\in\delta\; f(d)\le d'\}$. In fact $\mathscr{I}$ and $\mathscr{F}$ are 2-functors, since if $f\le g:D\to D'$, then $\mathscr{I}f\le\mathscr{I}g$ and $\mathscr{F}f\le\mathscr{F}g$. With these definitions, $\downarrow_D : D\to\mathscr{I}(D)$ and $\uparrow_D : D\to\mathscr{F}(D)$ are natural in D.

Now if $f: D \to D'$ is a morphism of distributive lattices then so are $\mathscr{I}f$ and $\mathscr{F}f$; and taking inverse images under f gives maps

$$f^{-1}: \mathscr{I}(D') \to \mathscr{I}(D) \quad \text{and} \quad f^{-1}: \mathscr{F}(D') \to \mathscr{F}(D)$$

which are right and left adjoints to $\mathscr{I}f$ and $\mathscr{F}f$ respectively. Thus in particular $\mathscr{I}f$ is a morphism of frames. The following result, whilst easily proved, provides the key that unlocks the door between Heyting algebras and open maps of locales:

2.2. Proposition. (i) *If $f: A \to B$ is a morphism of Heyting algebras, then the left adjoint f^{-1} of $\mathscr{F}f$ satisfies Frobenius reciprocity.*

(ii) *Suppose that $f: D \to D'$ is an order-preserving map between distributive lattices which has a left adjoint $f_!$ satisfying Frobenius reciprocity. Then $\mathscr{I}(f_!)$ is left adjoint to $\mathscr{I}f$ and also satisfies Frobenius reciprocity.* (*Similarly for $\mathscr{F}(f_!)$.*)

Proof. (i) Suppose that $\alpha \in \mathscr{F}(A)$ and $\beta \in \mathscr{F}(B)$. Since f^{-1} is left adjoint to $\mathscr{F}f$, we always have $f^{-1}(\beta \wedge \mathscr{F}f(\alpha)) \leq f^{-1}(\beta) \wedge \alpha$. We have to show conversely that

$$f^{-1}(\beta \wedge \mathscr{F}f(\alpha)) \subseteq f^{-1}(\beta) \wedge \alpha.$$

The meet of two filters α_1, α_2 in $\mathscr{F}(A)$ is $\alpha_1 \wedge \alpha_2 = \{a_1 \wedge a_2 \mid a_i \in \alpha_i\}$. So if $a \in f^{-1}(\beta \wedge \mathscr{F}f(\alpha))$, then there are $b \in \beta$ and $a' \in \alpha$ with $b \wedge f(a') \leq f(a)$. Hence $f(a' \to a) \geq b \in \beta$, so that $a' \to a \in f^{-1}(\beta)$. Then

$$a \geq (a' \to a) \wedge a' \in f^{-1}(\beta) \wedge \alpha$$

and thus $a \in f^{-1}(\beta) \wedge \alpha$, as required.

(ii) Since $\mathscr{I}$ is a 2-functor, it is automatic that $\mathscr{I}(f_!)$ is left adjoint to $\mathscr{I}f$. Given $I \in \mathscr{I}(D)$ and $I' \in \mathscr{I}(D')$, suppose that $d \in \mathscr{I}f_!(I') \wedge I$. Since the meet of two ideals in $\mathscr{I}(D)$ is given by their intersection, we have that $d \in \mathscr{I}f_!(I')$ and $d \in I$. So there is $d' \in I'$ with $d \leq f_!(d')$, and therefore

$$d = f_!(d') \wedge d = f_!(d' \wedge fd)$$

since $f_!$ satisfies Frobenius reciprocity. But $d' \wedge fd \in I' \wedge \mathscr{I}f(I)$, so $d \in \mathscr{I}f_!(I' \wedge \mathscr{I}f(I))$. Thus $\mathscr{I}f_!(I') \wedge I \subseteq \mathscr{I}f_!(I' \wedge \mathscr{I}f(I))$ and since $\mathscr{I}f_!$ is left adjoint to $\mathscr{I}f$, the reverse inclusion is immediate. □

If A is a Heyting algebra, $\mathscr{F}(A)$ is a distributive lattice and $\mathscr{I}(\mathscr{F}A)$ is a frame: regarding it as an object in **Loc**, let us write $\phi(A)$ for this *locale of ideals of filters* of A.

2.3. Theorem. *Taking the locale of ideals of filters gives a contravariant functor $\phi: \mathbf{Ha}^{\mathrm{op}} \to \mathbf{OLoc}$ from the category of Heyting algebras to the category of locales and open continuous maps. This functor takes monomorphisms in* **Ha** *to surjections in* **Loc**.

Moreover for each Heyting algebra A there is a monomorphism $i_A: A \rightarrowtail \phi(A)$ in **Ha** *which is natural in A.*

Proof. Given $f: A \to B$ in **Ha**, $\mathcal{F}f: \mathcal{F}(A) \to \mathcal{F}(B)$ is a morphism of distributive lattices and so $\mathcal{I}(\mathcal{F}f): \mathcal{I}(\mathcal{F}A) \to \mathcal{I}(\mathcal{F}B)$ is a morphism of frames. By Proposition 2.2(i), $\mathcal{F}f$ has a left adjoint f^{-1} satisfying Frobenius reciprocity; hence by Proposition 2.2(ii), $\mathcal{I}(f^{-1})$ is left adjoint to $\mathcal{I}(\mathcal{F}f)$ and also satisfies Frobenius reciprocity. Therefore we have an open continuous map of locales $\phi f: \phi B \to \phi A$ with

$$(\phi f)_! = \mathcal{I}(f^{-1}), \qquad (\phi f)^* = \mathcal{I}(\mathcal{F}f) \quad \text{and} \quad (\phi f)_* = (\mathcal{I}f)^{-1}.$$

Since $\mathcal{I}$ and $\mathcal{F}$ are functorial, so is ϕ. Also $\mathcal{I}$ and $\mathcal{F}$ preserve monomorphisms of distributive lattices; so if f is a monomorphism in **Ha**, $\mathcal{F}f$ and hence $(\phi f)^* = \mathcal{I}(\mathcal{F}f)$ are monomorphisms, and thus ϕf is a surjective map of locales.

Given a Heyting algebra A, define $i_A: A \to \phi A$ to be the composition of $\uparrow_A: A \rightarrowtail \mathcal{F}(A)$ with $\downarrow_{\mathcal{F}A}: \mathcal{F}(A) \rightarrowtail \mathcal{I}(\mathcal{F}A) = \phi A$. Then not only is i_A a monomorphism of distributive lattices but also by Lemma 2.1 it preserves implications, so that it is a morphism of Heyting algebras (despite the fact that $\mathcal{F}(A)$ is not a Heyting algebra). Since $\downarrow$ and $\uparrow$ are natural, given $f: A \to B$ in **Ha** we have

$$(\phi f)^* \circ i_A = i_B \circ f. \qquad \square$$

Remark. Every element I of ϕA is expressible as a join of meets of elements from A, viz.

$$I = \bigvee_{\alpha \in I} \bigwedge_{a \in \alpha} i_A(a).$$

However given $f: A \to B$ in **Ha** with B a locale, we cannot necessarily extend f along i_A to the inverse image part of an open continuous map of locales $\bar{f}: B \to \phi A$. For example taking $f = \mathrm{id}_B$, $\bar{f}^*: \phi B \to B$ would have to be given by

$$\bar{f}^*(I) = \bigvee \{ \bigwedge \beta \mid \beta \in I \}.$$

But since *arbitrary* meets do not generally distribute over joins in B, this formula does not give a join-preserving map. In particular ϕ is not right adjoint to the forgetful functor $\mathbf{OLoc} \to \mathbf{Ha}^{\mathrm{op}}$.

3. Interpolation for Heyting algebras

Theorem A. *Monomorphisms are stable under pushout in* **Ha**.

Proof. Suppose we have $f: A \to B$ and $g: A \to C$ in **Ha**. Applying the functor ϕ of Theorem 2.3, let

$$\begin{array}{ccc} P & \xrightarrow{q} & \phi C \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \phi g} \\ \phi B & \xrightarrow{\phi f} & \phi A \end{array}$$

be a pullback square in **Loc**. By Theorem 1.1, p and q are both open since ϕf and ϕg are. Hence there is a square of morphisms in **Ha**

$$\begin{array}{ccccc} B & \xrightarrow{i_B} & \phi B & \xrightarrow{p^*} & P \\ \uparrow{\scriptstyle f} & & & & \uparrow{\scriptstyle q^*} \\ & & & & \phi C \\ & & & & \uparrow{\scriptstyle i_C} \\ A & & \xrightarrow{\quad g \quad} & & C \end{array}$$

which commutes since i is natural. Now if f is a monomorphism, ϕf is a surjection and then by Theorem 1.1, so is q; therefore $q^* \circ i_C$ is a monomorphism. But the pushout of f along g factors through $q^* \circ i_C$, so that pushout is also a monomorphism. □

Recall that congruences on a Heyting algebra A are in correspondence with filters on A: given $\alpha \in \mathcal{F}(A)$, we get a congruence by defining

$$a \sim a' \Leftrightarrow a \leftrightarrow a' \in \alpha.$$

Let $A \to A/\alpha$ denote the quotient of A by α. We need some simple facts about image factorizations and pushouts of quotients in **Ha**.

3.1. Lemma. *Suppose that $f: A \to B$ is a morphism of Heyting algebras.*

(i) *If $\beta \in \mathcal{F}(B)$, then the factorization of $A \xrightarrow{f} B \longrightarrow B/\beta$ through $A \to A/f^{-1}\beta$ is a monomorphism:*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \downarrow \\ A/f^{-1}\beta & \overset{\bar{f}}{\rightarrowtail} & B/\beta \end{array}$$

(ii) *If $\alpha \in \mathcal{F}(A)$, then the pushout along f of the quotient of A by α is the quotient of B by $\mathcal{F}f(\alpha)$, i.e. there is a pushout square*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \downarrow \\ A/\alpha & \xrightarrow{\quad \bar{f} \quad} & B/\mathcal{F}f(\alpha). \end{array}$$

□

Theorem B. *Every pushout square in* **Ha** *has the interpolation property* (*cf. the Introduction*).

Proof. Let

$$\begin{array}{ccc} B & \xrightarrow{\ h\ } & P \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle k} \\ A & \xrightarrow{\ g\ } & C \end{array}$$

be a pushout square in **Ha**. It follows from Lemma 3.1 that for any $\beta \in \mathcal{F}(B)$, $C \xrightarrow{k} P \longrightarrow P/\mathcal{F}h(\beta)$ factors as

$$C \longrightarrow C/\mathcal{F}g(f^{-1}\beta) \xrightarrow{\bar{k}} P/\mathcal{F}h(\beta)$$

and that

$$\begin{array}{ccc} B/\beta & \xrightarrow{\ \hat{h}\ } & P/\mathcal{F}h(\beta) \\ \uparrow{\scriptstyle \bar{f}} & & \uparrow{\scriptstyle \bar{k}} \\ A/f^{-1}\beta & \xrightarrow{\ \hat{g}\ } & C/\mathcal{F}g(f^{-1}\beta) \end{array}$$

is also a pushout square. But since $\bar{f}$ is a monomorphism, by Theorem A so is $\bar{k}$: hence $k^{-1}(\mathcal{F}h(\beta)) = \mathcal{F}g(f^{-1}\beta)$.

Now if $b \in B$, $c \in C$ and $h(b) \leq k(c)$, taking $\beta = \uparrow_B(b)$, we have $kc \in \uparrow_P(hb) = \mathcal{F}h(\beta)$, so that $c \in k^{-1}(\mathcal{F}h(\beta)) = \mathcal{F}g(f^{-1}\beta)$. Hence there is $a \in A$ with $b \leq f(a)$ and $g(a) \leq c$, as required. □

Remark. The first part of the proof of Theorem B is really just the (dual of the) proof, familiar in the context of regular categories, that stability of image factorizations under pullback implies the Beck–Chevalley condition for existential quantification (cf. 3.2.1 of [3]). The last part of the proof is thus a particular instance of the fact remarked upon in the Introduction, that under suitable circumstances the interpolation property is equivalent to a Beck–Chevalley condition.

4. The situation for distributive lattices and frames

In conclusion, we make some remarks about the analogues of Theorems A and B for the category **Dl** of distributive lattices and the category **Frm** of frames.

(a) *Not every pushout square in* **Dl** *has the interpolation property.* For example, let $d \in D$ be an element of a distributive lattice which does not have a complement in D. Let $f : D \to \uparrow_D(d)$ and $g : D \to \downarrow_D(d)$ be the morphisms defined by

$$f(x) = d \vee x \quad \text{and} \quad g(x) = d \wedge x.$$

Then the pushout of f along g in **Dl** is the trivial lattice **1** (in which $\bot = \top$):

$$\begin{array}{ccc} \uparrow_D(d) & \xrightarrow{\;h\;} & \mathbf{1} \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle k} \\ D & \xrightarrow{\;g\;} & \downarrow_D(d) \end{array}$$

Now $h(\top) \leq k(\bot)$ in **1**, but if there were an $x \in D$ with $\top \leq f(x)$ and $g(x) \leq \bot$, we should have $\top = d \vee x$, $d \wedge x = \bot$, i.e. d would be complemented, contrary to assumption.

(b) Applying the functor $\mathscr{I}$ (which is left adjoint to the forgetful functor **Frm**$\to$**Dl**) to the square in (a), we obtain a pushout square in **Frm** for which the interpolation property fails. (There are many others.)

(c) *Monomorphisms are stable under pushout in* **Dl**. One way of proving this (constructively) is to use Theorem A together with the fact that the left adjoint of the inclusion of the full subcategory of Boolean algebras into **Dl** preserves monomorphisms and the unit of the adjunction is a monomorphism.

(d) As is well known, surjections are not stable under pullback in **Loc**, so that the analogue of Theorem A fails for **Frm**. Indeed **Frm** *fails to have the amalgamation property*. For example[1], let $X = \mathbb{N} \cup \{\infty\}$ with topology:

$$U \subseteq X \text{ is open} \Leftrightarrow U = \emptyset \text{ or } X \setminus U \text{ is a finite subset of } \mathbb{N}.$$

Putting the discrete topology on $\mathbb{N}$, let $i : \mathbb{N} \hookrightarrow X$ denote the inclusion regarded as a continuous map. The corresponding frame morphism $i^* : \Omega(X) \to P(\mathbb{N})$ between the lattices of open sets is actually a monomorphism. But the pushout of i^* along the monomorphism $\Omega(X) \hookrightarrow P(X)$ is

$$\begin{array}{ccc} P(\mathbb{N}) & \xrightarrow{\;\mathrm{id}\;} & P(\mathbb{N}) \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^{-1}} \\ \Omega(X) & \hookrightarrow & P(X) \end{array}$$

and $i^{-1} : P(X) \to P(\mathbb{N})$ is not a monomorphism.

[1] I am grateful to P.T. Johnstone for suggesting this example.

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